

# A New Modified Newton Method use of Haar wavelet for solving Nonlinear equations

Bijaya Mishra\*, Ambit Kumar Pany †, Salila Dutta ‡

January 3, 2017

## Abstract

In this paper, we present a new modified Newton method a use of Haar wavelet formula for solving non-linear equations. This new method do not require the use of the second-order derivative. It is shown that the new method has third-order of convergent. Furthermore, some numerical experiments are conducted which confirm our theoretical findings.

**Keywords:** *Newton method; Haar wavelet; Iterative Method; Third-order convergence; Non-linear equations; Root-finding*

## 1 Introduction

In numerical analysis, finding a solution of non-linear equation is one of the most attractive problem. In this paper, we emphasize on an iterative method to find a simple root  $\alpha$  of a non-linear equation  $f(x) = 0$ , i.e.,  $f(\alpha) = 0$  and  $f'(\alpha) \neq 0$ . Here, we less concern about multiple roots. Newtons method [1],[6] is the well known algorithm to solve nonlinear equation. It is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad n = 0, 1, 2, \dots \quad (1.1)$$

and it converges quadratically.

Earlier, [3]-[5] and [8]-[13] derived third-order convergence methods based on integral interpretation of Newton's method. Where, Newton's method derived from different quadrature formulas for the indefinite integral arising from Newton's theorem [2]

$$f(x) = f(x_n) + \int_{x_n}^x f'(t)dt. \quad (1.2)$$

---

\*Department of mathematics, Gandhi Institute for Technological Advancement, Bhubaneswar-752054, India. Email: bijayamishra.math@gmail.com

†Center for Applied Mathematics, Siksha O Anusandhan University, Bhubaneswar, Odisha, India, 751030. Email: ambit.pany@gmail.com

‡Department of Mathematics, Utkal University, Bhubaneswar, Odisha, India, 751004. Email: saliladutta516@gmail.com

Weerakoon et al.[3] have approximated the integral part of (1.2) by trapezoidal rules and derived a variant of Newton's method. It is, further, shown that this method converges cubically. Subsequently, Frontini et al. [4] have proposed a third order convergent method by approximate the integral by the midpoint rule. In [10], Homeier has developed a cubically convergent iteration scheme by considering Newtons theorem for the inverse function. Further, in [11], [12] modified Newton methods are derived for multivariate case. Kou et al. in [13] have applied a new interval of integration on Newtons theorem and arrived a third-order convergent iterative scheme.

Recently, Islam et al.[7] have applied Haar wavelet function to derived quadrature rules for indefinite integration. In this paper, we modified Newton's theorem by using the quadrature rule proposed by Islam in [7]. It is shown that the new method has third order convergent. Further, the new method did not evaluate second derivative of  $f$ . The efficiency of the new method is demonstrated by numerical examples.

This paper is organized as follows. In Section 2, we discuss a modified Newton's method. Section 3, we establish convergence analysis for the new method. finally in section 4, various numerical experiments conduct to confirm our theoretical finding.

## 2 A Modified Newton's Method

To derive the new method, we consider Newtons theorem

$$f(x) = f(x_n) + \int_{x_n}^x f'(\tau) d\tau \quad (2.1)$$

We use the Haar wavelet function to approximate the integral term of (2.1) as

$$\int_{x_n}^x f'(\tau) d\tau = \frac{(x - x_n)}{2M} \sum_{k=1}^{2M} f'\left(x_n + \frac{(x - x_n)(k - 0.5)}{2M}\right). \quad (2.2)$$

where  $M = 2^{J_1}$  and  $J_1$  is the maximum level of resolution of Haar wavelets, see [7].

Substitute (2.2) in (2.1) to obtain

$$f(x) = f(x_n) + \frac{(x - x_n)}{2M} \sum_{k=1}^{2M} f'\left(x_n + \frac{(x - x_n)(k - 0.5)}{2M}\right). \quad (2.3)$$

Now, looking for  $f(x) = 0$  we arrive at

$$x_{n+1} = x_n - \frac{2M(f(x_n))}{\sum_{k=1}^{2M} f'\left(x_n + \frac{(x - x_n)(k - 0.5)}{2M}\right)} \quad (2.4)$$

Further, substitute  $x_{n+1} = x$  in (1.1) and replace  $x - x_n$  in (2.4) we obtain the new method as

$$x_{n+1} = x_n - \frac{2M(f(x_n))}{\sum_{k=1}^{2M} f'\left(x_n - \frac{f(x_n)}{f'(x_n)} \frac{(k - 0.5)}{2M}\right)} \quad (2.5)$$

### 3 Convergence Analysis

**Theorem 3.1.** *Let the function  $f : \mathbb{D} \subset \mathbb{R} \rightarrow \mathbb{R}$  has a simple root  $\alpha \in \mathbb{D}$ , where  $\mathbb{D}$  is an open interval. Assume  $f(x)$  has first, second and third derivatives in the interval  $\mathbb{D}$ . If the initial guess  $x_0$  is closed to  $\alpha$ , then the method defined by (2.5) converges cubically to  $\alpha$ .*

*Proof.* Let  $\alpha$  is the simple root of  $f(x)$  and  $x_n = \alpha + e_n$ . A use of Taylor expansion with  $f(\alpha) = 0$ , we have

$$f(x_n) = f'(\alpha)(e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4)), \quad (3.6)$$

where  $C_k = \frac{1}{k!} \frac{f^{(k)}(\alpha)}{f'(\alpha)}$ . Again

$$f'(x_n) = f'(\alpha)(1 + 2C_2 e_n + 3C_3 e_n^2 + 4C_4 e_n^3 + O(e_n^4)). \quad (3.7)$$

Further, dividing (3.6) by (3.7) yields

$$\frac{f(x_n)}{f'(x_n)} = (e_n - C_2 e_n^2 + 2(C_2^2 - C_3)e_n^3 + O(e_n^4)). \quad (3.8)$$

Now,

$$x_n - M_k \frac{f(x_n)}{f'(x_n)} = x_n - M_k (e_n - C_2 e_n^2 + 2(C_2^2 - C_3)e_n^3 + O(e_n^4)), \quad (3.9)$$

where  $M_k = \frac{(k-0.5)}{2M}$ .

Equation (3.9) rewrite as

$$\begin{aligned} x_n - M_k \frac{f(x_n)}{f'(x_n)} &= x_n - M_k (e_n - C_2 e_n^2 + 2(C_2^2 - C_3)e_n^3 + O(e_n^4)), \\ &= x_n - M_k e_n + M_k C_2 e_n^2 - 2(C_2^2 - C_3)M_k e_n^3 + O(e_n^4), \\ &= x_n - (1 - 1 + M_k)e_n + M_k C_2 e_n^2 - 2(C_2^2 - C_3)M_k e_n^3 + O(e_n^4), \\ &= x_n - e_n + (1 - M_k)e_n + M_k C_2 e_n^2 - 2(C_2^2 - C_3)M_k e_n^3 + O(e_n^4), \\ &= \alpha + (1 - M_k)e_n + M_k C_2 e_n^2 - 2(C_2^2 - C_3)M_k e_n^3 + O(e_n^4). \end{aligned} \quad (3.10)$$

From (3.10) we can easily find that

$$f'(x_n - M_k \frac{f(x_n)}{f'(x_n)}) = f'(\alpha)(1 + 2C_2(1 - M_k)e_n + (2C_2^2 M_k + 3C_3(1 - M_k)^2)e_n^2 + O(e_n^3)). \quad (3.11)$$

Hence,

$$\begin{aligned}
\sum_{k=1}^N f'(x_n - M_k \frac{f(x_n)}{f'(x_n)}) &= \sum_{k=1}^N f'(\alpha) (1 + 2C_2(1 - M_k)e_n + (2C_2^2 M_k + 3C_3(1 - M_k)^2)e_n^2 + O(e_n^3)), \\
&= f'(\alpha) \sum_{k=1}^N (1 + 2C_2(1 - M_k)e_n + (2C_2^2 M_k + 3C_3(1 - M_k)^2)e_n^2 + O(e_n^3)), \\
&= f'(\alpha) (\sum_{k=1}^N 1 + 2C_2 e_n \sum_{k=1}^N (1 - M_k) + 2C_2^2 e_n^2 \sum_{k=1}^N M_k + 3C_3 e_n^2 \sum_{k=1}^N (1 - M_k)^2 + O(e_n^3)), \\
&= f'(\alpha) (N + 2C_2 N e_n - 2C_2 e_n N/2 + 2C_2^2 e_n^2 N/2 + 3C_3 e_n^2 N \\
&\quad + 3C_3 e_n^2 (\frac{N}{3} - \frac{1}{12N}) - 6C_3 e_n^2 N/2 + O(e_n^3)) \\
&= f'(\alpha) (N + 2C_2 N e_n + (NC_2^2 + NC_3 - \frac{C_3}{4N})e_n^2 + O(e_n^3)) \tag{3.12}
\end{aligned}$$

substitute (3.12) and (3.6) in (2.5) we obtain

$$\begin{aligned}
x_{n+1} &= x_n - \frac{(e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4))}{(N + 2C_2 N e_n + (NC_2^2 + NC_3 - \frac{C_3}{4N})e_n^2 + O(e_n^3))}, \\
&= x_n - (e_n + (-C_2^2 + \frac{C_3}{4N^2})e_n^3 + O(e_n^4)). \tag{3.13}
\end{aligned}$$

Subtract  $\alpha$  from both side of (3.13), then we have

$$e_{n+1} = (C_2^2 - \frac{C_3}{4N^2})e_n^3 + O(e_n^4) \tag{3.14}$$

this completes the rest of the proof.  $\square$

## 4 Numerical Examples

In this section, we present some numerical results for various third order convergent iterative methods.

The following methods were compared:

Modified Newton's Method in Weerakoon and Fernando [3]  $x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n - \frac{f(x_n)}{f'(x_n)}) + f'(x_n)}$ .

Modified Newton's Method proposed by Frontini et al. [4]  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n - 2\frac{f(x_n)}{f'(x_n)})}$ .

Modified Newton's Method derived by Ozbal et al. [5]  $x_{n+1} = x_n - \frac{f(x_n)}{2} (\frac{1}{f'(x_n)} + \frac{1}{f'(x_n - \frac{f(x_n)}{f'(x_n)})})$ .

Modified Newton's Method derived by Kou et al. [13]  $x_{n+1} = x_n - \frac{f(x_n + \frac{f(x_n)}{f'(x_n)}) - f(x_n)}{f'(x_n)}$ .

and method proposed by (2.5) with  $M = 1$ ,  $x_{n+1} = x_n - \frac{2(f(x_n))}{\sum_{k=1}^2 f'(x_n - \frac{f(x_n)}{f'(x_n)} \frac{(k-0.5)}{2})}$ .

For every problem an attempt made to find an approximation  $x_n$  of the simple root of equation  $f(x) = 0$  through  $n$  times iterations. The number of function evaluations (NFE) is counted as the sum of the number of evaluations of the function  $f$  and its first order derivative  $f'$ . The computational results are displayed in Table 1.

The numerical experiments carried over the following equation:

$$\begin{aligned}
f_1(x) &= x^5 - x + 1, \\
f_2(x) &= \cos x - x, \\
f_3(x) &= \arctan x, \\
f_4(x) &= 10xe^{-x^2} - 1, \\
f_5(x) &= e^{-x}\sin x + \log(x^2 + 1), \\
f_6(x) &= x^3 - e - x, \\
f_7(x) &= e - x - \cos x,
\end{aligned}
\tag{4.15}$$

The numerical results presented in Table 1 show that the proposed method has perform equally as compared with the other methods of the same order. Thus, the new methods can compete with other third-order methods in literature.

## 5 Conclusion

This article deals with a new modified Newton methods for solving nonlinear equations. In Theorem (3.1), it is proved that the new method has third order convergence. Further, the new methods can compete with other third-order methods. Finally, numerical experiments are conducted to confirm our theoretical findings. The new method has great practical utility.

## References

- [1] A.M. Ostrowski, *Solution of Equations in Euclidean and Banach Space*, third ed., Academic Press, New York, 1973.
- [2] J.E. Dennis, R.B. Schnable, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*, Prentice Hall, 1983.
- [3] S. Weerakoon, T.G.I. Fernando, *A variant of Newtons method with accelerated third-order convergence*, Appl. Math. Lett., 13 (2000), pp. 87–93.
- [4] M. Frontini, E. Sormani, *Some variants of Newtons method with third-order convergence*, Appl. Math. Comput., 140 , (2003), pp. 419–426.
- [5] A. Y. Ozban, *Some new variants of Newtons method*, Appl. Math. Lett., 17 ,(2004), pp. 677–682.
- [6] Conte, S. D. and De Boor Carl, *Elementary numerical analysis*, Mc Graw Hill, Kogakusha Ltd., (1972).

- [7] S. u. Islam, I. Aziz and F. Haq, A comparative study of numerical integration based on haar wavelets and hybrid functions, *Comput. Math. Appl.*, 59, (2010), pp. 2026–2036
- [8] M. Frontini, E. Sormani, *Modified Newtons method with third-order convergence and multiple roots*, *J. Comput. Appl. Math.*, 156, (2003), pp. 345–354.
- [9] M. Frontini, E. Sormani, *Third-order methods from quadrature formulae for solving systems of nonlinear equations*, *Appl. Math. Comput.*, 149, (2004), pp. 771–782.
- [10] H.H.H. Homeier, A modified Newton method for root finding with cubic convergence, *J. Comput. Appl. Math.*, 157, (2003), pp. 227–230.
- [11] H.H.H. Homeier, *A modified Newton method with cubic convergence: the multivariate case*, *J. Comput. Appl. Math.*, 169, (2004), pp. 161–169.
- [12] H.H.H. Homeier, *On Newton-type methods with cubic convergence*, *J. Comput. Appl. Math.*, 176, (2005), pp. 425–432.
- [13] J. Kou, Y. Li, X. Wang, *A modification of Newton method with third-order convergence*, *Appl. Math. Comput.*, 181, (2006), pp. 1106–1111.

Function	$x_0$	Various Method	IT	NFE	$x_n$
$f_1$	2	MNM([3])			Diverse
		MNM([4])			Diverse
		MNM([5])	13	39	-1.16730397826142
		MNM([13])	17	51	-1.16730397826142
		MNM New	9	36	-1.16730397826142
$f_2$	1.2	MNM([3])	4	12	0.739085133215161
		MNM([4])			Diverse
		MNM([5])			Diverse
		MNM([13])	4	12	0.739085133215161
		MNM New	4	16	0.739085133215161
$f_3$	3	MNM([3])			Diverse
		MNM([4])			Diverse
		MNM([5])			Diverse
		MNM([13])	4	12	0.0000000000000015
		MNM New	4	16	0.0
$f_4$	2.5	MNM([3])	4	12	1.67963061042845
		MNM([4])	7	21	1.67963061042845
		MNM([5])	4	12	0.101025848315685
		MNM([13])	6	18	1.67963061042845
		MNM New	5	20	1.67963061042845
$f_5$	1.3	MNM([3])	3	9	7.68481808334733E-021
		MNM([4])	4	12	1.7197167733818E-028
		MNM([5])	4	12	5.12759588393657E-030
		MNM([13])	3	9	3.82180552357862E-019
		MNM New	3	12	4.53468286561001E-017
$f_6$	2	MNM([3])	5	15	0.77288295914921
		MNM([4])	4	12	0.77288295914921
		MNM([5])	4	12	0.77288295914921
		MNM([13])	5	15	0.77288295914921
		MNM New	4	16	0.77288295914921
$f_7$	2	MNM([3])	3	9	1.29269571937339
		MNM([4])	3	9	1.29269571937339
		MNM([5])	4	12	1.29269571937339
		MNM([13])	4	12	1.29269571937339
		MNM New	3	12	1.29269571937339

Table 1: Comparison of various third order convergent iterative methods and the New Newton method